Journal of Basic and Applied Engineering Research
p-ISSN: 2350-0077; e-ISSN: 2350-0255; Volume 5, Issue 3; January-March, 2018, pp. 162-172

http://www.krishisanskriti.org/Publication.html

A Review on Characterizations of Geometric and Exponential Distribution

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Abstract—*The characterization of Geometric and Exponential distribution have a lot of application in Actuarial, Biological Sciences, Engineering problems, Queuing theory etc. The characterization of geometric and exponential distribution have been recently given by many authors based on Order Statistics, Independence of the random variables, characteristic functional equations etc. This paper is not enough to review all the characterizations of Geometric and Exponential distribution and I have reviewed some of the characterizations.*

1. INTRODUCTION

The characterization of exponential and geometric distributions have a lot of applications in Actuarial, biological sciences, engineering problems, queuing theory etc. The exponential and geometric distributions are remarkably friendly. So everyone introducing a new concept to classify distribution inevitably uses the exponential distribution as their concept .The characterization of the exponential and geometric have been recently given by many authors based on Order Statistics, Independence of the Random Variables, Record Values, etc. A paper is inadequate to catalogue the enormous number of characterizations of both exponential and geometric distribution. All of them cannot be included in this paper. So, I have reviewed only some characterizations of exponential, and geometric distributions.

2. REVIEW OF LITERATURE

Desu [1971] showed that the exponential distribution is the only one with the property that for all k, k times the minimum of the random sample of size k from the distribution has the same distribution as a single observation from the distribution.

Puri and Rubin [1970] proved that if X_1 and X_2 are independent copies of a random variable X with density f, then X and $\left|X_1 - X_2\right|$ have the same distribution if and only if f is exponential distribution.

Arnold and Ghosh [1976] showed that if $X_{2:2} - X_{1:2}$ given $X_{2:2} - X_{1:2} > 0$ and $X_{1:1}$ has the same distribution then the X_i 's are geometrically distributed.

H.J.Rossberg [66] has reported that the condition that $\sum_{j=k}^{n} C_j X_j$ (with $\sum_{j=k}^{n} C_j = 0$ and C_j 's not all zero) and X_h' (h ≤ k) are independent ensure that the distribution is exponential form. If further $P[X_k' - X_{k-1}' \geq x] = [1 - F_X(x)]^{n-k+1}$ then the start of the distribution is at zero.

Gordon B. Growford [1966] has shown that if X and Y are non – degenerate

Independent random variables with X – Y and min(X, Y) are independent then for some constant a, (X – a) and (Y – a) have exponential distribution function.

A.P.Basu[1] shown that if X_1 and X_2 have the same absolutely continuous distribution function F(x) iff the random variables X_1 and $W = X_2 - X_1$ are independently distributed.

Ursula Gather [1998] obtained a characterization of the exponential distribution by properties of order statistics.

El – Neweihi and Govindarajulu [1978] showed that if $X_{1:n}$ is independent of

$X_{k:n} - X_{1:n} = 0$ for some k >1, then the X's are geometric (or degenerate).

Ashok K.Nanda [2010] obtained a characterization of distributions through failure rate and mean residual life function.

3. CHARECTERIZATIONS OF GEOMETRIC AND EXPONENTIAL DISTRIBUTION BASED ON PROPERTIES OF ORDERS STATISTICS

Some Characterizations of the geometric distribution are given by R.C. Srivastava (1974), Barry C.ARNOLD (1979) and Z. Govindarajulu (1980) among others. Almost all these characterizations are on the results of the order statistics and the lack of memory property. In this chapter, I would like to review their works about the characterizations of Geometric distribution. And also, I have reviewed the characterizations of exponential distribution based on order statistics, which were given by M.S. Srivastava, M.Ashanullah and Sukhatme.

3.1.1. Introduction

In this note, we discuss two simple properties of the geometric distribution which was given by R.C.Srivastava and show that very weak forms of these properties are characterizing properties of the geometric distribution. Let $X_1, X_2, \ldots\ldots X_n$, $n \geq 2$ be n independent observations on a random variable X having a geometric distribution with density.

$$P_j = p\,(1-p)^{(j-\alpha)/\beta},\ j= \alpha,\alpha+\beta,\alpha+2\beta\ldots\ldots \quad (3.1.1a)$$

The parameters α and β are respectively location and scale parameters and the parameter p is the geometric parameter. Also the $Y_1 \leq Y_2 \leq \ldots \leq Y_n$ denote the corresponding order statistics .Write $Z = \sum_{j=2}^{n} (Y_j - Y_1)$. It is well known that Y_1 and Z are independent.

For formulating the second property, let us consider two independent random variables X_1 and X_2 having geometric distributions with the same scale and location parameters α and β but different geometric parameters p_1 and p_2.It is known that

$U = \min(X_1, X_2)$ and $V=X_2-X_1$ are independently distributed.

These properties enable us to obtain two characterizations of the geometric distribution as given in the following theorems. Throughout this article, we take α = 0, β = 1. However, it is clear that the conclusions are true for all admissible values of α and β.

Theorem 1

Let $X_1, X_2, \ldots\ldots X_n$ be iid r.v's with distribution function F(x) with positive mass at α, α +β, α +2β……and Let $Y_1 \leq Y_2 \leq \ldots \leq Y_n$ be the orders statistics. Then.

$$P(Y_1= \alpha+j\beta, Z=0) = P(Y_1= \alpha +j\beta).P(Z=0).$$

$$\text{for all } j = 0,1,2.... \quad (3.1.1b)$$

iff, F(x) is geometric distribution with density given by (3.1.1a).

Proof:

From (3.1.1b), we have

$$P (Y_1 = j, Z = 0) = P(Y_1=j)\, P(Z=0)$$

$$= \prod_{i=1}^{n} P(X_i=j)$$

$$=p_j^{\,n} \quad (3.1.1c)$$

and $P(Y_1=j)\ P(Z\neq 0) = P(Y_1=j, Z\neq 0)$

$$= \sum_{r=1}^{n-1}\binom{n}{r} p_j^r \left(\sum_{k=j+1}^{\infty} p_k\right)^{n-r} \quad (3.1.1d)$$

Substituting the value of $P(Y_1=j)$ from (3.1.1c) into (3.1.1d), we have,

$$\frac{p_j^{\ n}}{P(Z=0)}.P(Z\neq 0) = \sum_{r=1}^{n-1}\binom{n}{r} p_j^{\ r}\left(\sum_{k=j+1}^{\infty} p_k\right)^{n-r}$$

$$\sum_{r=1}^{n-1}\binom{n}{r} p_j^{\ r}\left(\sum_{k=j+1}^{\infty} p_k\right)^{n-r} = A.\ p_j^{\,n} \quad (3.1.1e)$$

Where $A = \dfrac{P(Z\neq 0)}{P(Z=0)}$

Equation (2.1.1e) can now be written as

$$\sum_{r=1}^{n-1}\binom{n}{r} b_j^{\ r} = A \quad (3.1.1f)$$

Since (3.1.1f) is true for all values of j, we also have,

$$\sum_{r=1}^{n-1}\binom{n}{r} b_{j+1}^{\ \ r} = A \quad (3.1..1g)$$

Subtracting (3.1.1g) from (3.1.1f), we have,

$$\sum_{r=1}^{n-1}\binom{n}{r}(b_j^{\ r} - b_{j+1}^{\ \ r}) = 0 \quad \text{or}$$

$$(b_j - b_{j+1}).\left[\sum_{r=1}^{n-1}(b_j^{\ r-1} + b_j^{\ r-2}.b_{j+1}\ldots\ldots\ldots + b_j^{\ r-1})\right] = 0.$$

(3.1.1h)

Since the quantity inside the square bracket is not equal to zero. We have $b_{j+1} = b_j$ for all j.

So we have $b_j = b_0$ or

$$p_j = \tfrac{a_j}{a_0}.p_0 \quad = (a_{j-1} - p_j).\frac{p_0}{a_0}$$

$$= a_{j-1}\frac{p_0}{a_0} - p_j\frac{p_0}{a_0}.$$

$$\text{Or}\quad p_j\left(1+\frac{p_0}{a_0}\right) = a_{j-1}\frac{p_0}{a_0}$$

$$p_j \ = \ a_{j-1}.p_0 = (a_{j-2} - p_{j-1})p_0$$

$= p_{j-1}(1-p_0)$

By induction, $p_j = (1-p_0)^j \cdot p_0$. This completes the proof .An immediate consequence of theorem 1 is,

Corollary 1:-

Let $S^2 = \sum (X_i - \overline{X})^2$. ThenP $(Y_1=j, S^2 = 0) = P(Y_1=j)\, P(S^2=0)$ for j = 0,1,….. iff F(x) is a geometric distribution.

Corollary 2:

F(x) is a geometric distribution iff

$P(Y_1= j, Y_k-Y_1 = 0, K = 2\ldots n) = P(Y_1=j).\ P(Y_k-Y_1=0, K=2\ldots n)$ for all j.

Theorem 2:-

Let X_1 and X_2 be independent random variables possibly with different distributions having positive mass at α, α + β, α + 2 β, …..

Then $P(U= \alpha + j\beta, V = l\beta) = P(U = \alpha + j\beta)\, P(V = l\beta)$ for l = 0,1 and all j (3.1.1i)

Proof:-

Let $P(X_1=j) = p_j$ and $P(X_2 = j) = q_j$, j = 0,1….. From (2.1.1i) we have,

$P(U=j)\, P(V=0) = P(U=j, V=0)$

$= P(X_1=j)\, P(X_2=j)$

$= p_j q_j$ (3.1.1j)

And $P(U = j)\, P(V=l) = P(U=j)\,(V=1)$

$= P(X_1 = j)\, P(X_2=j+1)$

$= p_j q_{j+1}$ (3.1.1k)

Also, $P(U = j) = p_j q_j + p_j \sum_{k=j+1}^{\infty} q_k + q_j \sum_{k=j+1}^{\infty} p_k$

Let $C_0 = P(V=0) = \sum_{j=0}^{\infty} p_j q_j > 0$

and $C_1 = P(V=1) = \sum_{j=0}^{\infty} p_j q_{j+1} > 0$.

Substituting these values in (2.1.1j) and (2.1.1k), we have,

$$p_j q_j = C_0 (p_j q_j + p_j \sum_{k=j+1}^{\infty} q_k + q_j \sum_{k=j+1}^{\infty} p_k) \quad (2.1.1l)$$

And

$$p_j q_{j+1} = C_1 (p_j q_j + p_j \sum_{k=j+1}^{\infty} q_k + q_j \sum_{k=j+1}^{\infty} p_k) \quad (3.1.1m)$$

Dividing (3.1.1m) by (3.1.1c), we obtain,

$$\frac{p_j q_{j+1}}{p_j q_j} = \frac{C_1}{C_0} \Rightarrow q_{j+1} = \frac{C_1}{C_0} q_j \qquad \text{for j=0,1,2.....}$$

Or

$$q_{j+1} = \left(\frac{C_1}{C_0}\right)^{j+1} q_j \quad (3.1.1n)$$

It follows that (3.1.1n) that X_2 has a geometric distribution. Substituting $q_j = \left(\frac{C_1}{C_0}\right)^{j} q_0$ in (3.1.1l), we

$$p_j \cdot \left(\frac{C_1}{C_0}\right)^{j} . q_0 = C_0 \left(p_j \left(\frac{C_1}{C_0}\right)^{j} + p_j \sum_{k=j+1}^{\infty} \left(\frac{C_1}{C_0}\right)^{k} . q_0 + \left(\frac{C_1}{C_0}\right)^{j} q_o \cdot \sum_{k=j+1}^{\infty} p_k \right)$$

Therefore $$p_j = \frac{C_0^{\ 2}}{C_0 - C_1} p_j + C_0 \sum_{k=j+1}^{\infty} p_k \quad (3.1.1O)$$

The equation is true for all values of j and also we have,

$$p_{j+1} = \frac{C_0^{\ 2}}{C_0 - C_1} p_j + C_0 \sum_{k=j+2}^{\infty} p_k \quad (3.1.1p)$$

Subtracting (3.1.1p) from (3.1.1O), we have,

$$p_j - p_{j+1} = \frac{C_0^{\ 2}}{C_0 - C_1} (p_j - p_{j+1}) + C_0 p_{j+1}$$

$$p_{j+1} \left(1 - \frac{C_0^{\ 2}}{C_0 - C_1} + C_0 \right) = p_j \left(1 - \frac{C_0^{\ 2}}{C_0 - C_1} \right)$$

$$p_{j+1} = p_j \left(\frac{C_0 - C_1 - C_0^{\ 2}}{C_0 - C_1 - C_0 C_1} \right) \quad (3.1.1q)$$

It is clear from (3.1.1q) that X_1 also has a geometric distribution.

3.1.2. Introduction

Here we discuss B.C. Arnold's characterization of Geometric distribution based on order statistics. Let $X_1, X_2, \ldots X_n$ be a set of positive integer valued random variable . We assume $0 < P(X_i = 1) < 1$. Denote the order statistics of the sample by $X_{1:n}, X_{2:n} \ldots X_{n:n}$. Geometric characterization based on distribution properties of the order statistics are not common . Those available assume a sample size 2. In particular, Arnold and Ghosh showed that if $X_{2:2} - X_{1:2}$ given $X_{2:2} - X_{1:2} > 0$ has the same distribution as $X_{1:1}$ then the X_i ' s are geometrically distributed . They conjectured that this result could be extended to tread the case of arbitrary sample size n

≥ 2 as follows. If for some k ($1 \leq k \leq n-1$), $X_{k+1:n} - X_{k:n}$ given $X_{k+1:n} - X_{k:n} > 0$ has the same distribution as $X_{1:n-k}$ then the X_i's are geometrically distributed. This conjecture is proved in the following section.

THE CHARACTERIZATIONS

Besides dealing with the geometric distribution, the two characterizations to be presented share another common feature. They are both consequences of a lemma recently proved by Shanbhag [1977].

Lemma 3.1.2:-

Let $\{(v_n.w_n), n = 0,1,2.....\}$ be a sequence of vectors with non – negative real components such that $v_n \neq 0$ for some $n \geq 1$ and $w_1 \neq 0$. Then

$$v_n = \sum_{n=0}^{\infty} v_{n+m}.w_n, \ m = 0,1,2.........$$

iff $\sum_{n=0}^{\infty} w_n b^n = 1$ and $v_n = v_0.b^n$, n = 1,2,....for some b>0.

Now suppose that $X_1, X_2, ...X_n$ be iid positive integer valued random variables with order statistics $X_{1:n}, X_{2:n} .... X_{n:n}$, for j = 1,2...... define $p_j = P(X_i = j)$ and $q_j = P(X_i > j)$ and assume $0 < p_1 < 1$.

Theorem 1:-

The conditional distribution of $X_{k+1:n} - X_{k:n}$ given $X_{k+1:n} - X_{k:n} > 0$ is the same as the unconditional distribution of $X_{1:n-k}$ for some pair (k,n), ($1 \leq k < n$), iff

$P(X_1 = j) = p.(1-p)^{j-1}$, j = 1,2...... for some $p \in (0,1)$

Proof:-

The if part is trivial. To verify the 'only if ' part, observe that for j = 1,2...., we have,

$$q_j^{n-k} = P(X_{1:n-k} > j) = P(X_{k+1:n} - X_{k:n} > j / X_{k+1:n} - X_{k:n} > 0)$$

$$= C. P(X_{k+1:n} - X_{k:n} > j)$$

$$= C. \sum_{l=1}^{\infty} P(X_{k:n=l,} X_{k+1:n} > l + j)$$

Now one can write,

$\{X_{k:n=l,} X_{k+1:n} > l + j\} = \bigcup_{m=1}^{k} E_m$, where E_m corresponds to the event described by exactly m X_i's assume the value l, k – m of the X_i's are less than l and n-k of the X_i's are greater than l+j. it follows that (using the convention $q_0 = 1$)

$$q_j^{n-k} = C \sum_{l=1}^{\infty} \sum_{m=1}^{k} \frac{n!}{m!.(k-m)!.(n-k)!}.p_l^{\,m} (1-q_{l-1})^{k-m.} q_{l+j}^{\ n-k}$$

$$= C \sum_{l=1}^{\infty} r_l q_{i+j}^{\ n-k}$$

Where $r_l = \sum_{l=1}^{\infty} \sum_{m=1}^{k} \frac{n!}{m!.(k-m)!.(n-k)!}.p_l^{\,m} (1-q_{l-1})^{k-m.}$

Now let $r_0 = 0$, $v_m = q_{m+1}^{\ n-k}$, $w_m = C_r^{\,m}$, m = 0,1,2...... The conditions of Shanbhag's are satisfied, since $0 < p_1 < 1$. It follows that,

$q_j^{n-k} = \alpha \beta^j$, j = 1,2..... and consequently $q_j = \alpha^{1/n-k.} (\beta^{1/n-k})^{j.}$

So that X_i's are indeed geometric random variables.

Theorem 2:-The random variable $X_{2:n} - X_{1:n}$ and the event $[X_{1:n} = 1]$ are independent and

$P(X_i = 1)$, $P(X_i = 2)$ and $P(X_i > 2)$ are strictly positive, iff

$P(X_1 = j) = p.(1-p)^{j-1}$, j = 1,2...... for some $p \in (0,1)$

PROOF:-

The if part is trivial. To show that the converse. Consider for j = 0,1,2....

$$P(X_{1:n} = 1, X_{2:n} - X_{1:n} > j) = P(X_{1:n} = 1, X_{2:n} > j+1)$$

$= r_1 q_{j+1}^{\ n-1}$

Where r_1 is as defined in the proof of the theorem 1 (with k = 1). Next observe that for j =0,1.2.....

$$P(X_{1:n} = 1) P(X_{2:n} - X_{1:n} > j) = C(n) \sum_{l=1}^{\infty} r_1 q_{j+1}^{\ n-1}$$

Again r_l is as defined in the above theorem (with k = 1). From the assumption of independence, we conclude that,

$$q_{j+1}^{\ n-1} = \widetilde{C} \sum_{l=1}^{\infty} r_l q_{l-1+j}^{\ n-1} \quad , j = 1,2........$$

Now let $v_m = q_{m+1}^{\ n-1}$ and $w_m = r_{m+1}$, m = 0,1,2....

The conditions of Shanbhags lemma are satisfied since $p_1 > 0, p_2 > 0$ and $p_1 + p_2 < 1$. It follows that, $q_j^{n-k} = \alpha \beta^j$, j = 1,2.......... and consequently,$q_j = \alpha^{1/n-1.} (\beta^{1/n-1})^{j.}$, j = 1,2........

So that X_i's are indeed geometric random variables.

3.13. Introduction

Here we discuss the characterization of geometric distribution based on the properties of order statistics, which was given by Z.Govindarajulu. Several contributions have been made to characterizing the geometric distribution based on order statistics. Recently

El – Neweihi and Govindarajulu.Z [1979] has characterized the geometric distribution using (i) the independence of X_{1n} and the event $\{X_{kn} = X_{1n}\}$ and (ii) the independence of X_{1n} and the event $\{X_{kn} - X_{1n} \in B\}$, where B = {m} or B = {m,m+1,..}.

They also characterize IFR or DFR property in terms of the monotonicity in i of P{ $X_{kn} – X_{1n}/ X_{1n} = a_i$}, where the

X_{in},(i = 1...n)denote order statistics in a random sample of size n and$a_1<a_2<$..... is the set of possible values of the underlying variable. It is of much interest to further generalize these results and Fergusons[1965,1967] results for independence.

3.1.3.1 Independent and Identically Distributed Components

Let X be a non – negative discrete random variable having {1,2...} for its set of possible values. Let $X_{1n} \leq .... \leq X_{nn}$ denote the order statistics in a random sample of size n drawn from the discrete population.Also, Let G(i) = P(X≥i) and q(i) = G(i+1)/G(i), i =1,2....

Theorem 3.1.3.1

For some arbitrarily fixed k (2 ≤ k ≤n) X_{1n} and the event {$X_{kn} - X_{1n} \geq m$} are independent for some m≥1 if and only if $G(i) = q^{i-1}$, for i = 1,2...provided $G(l) = q^{l-1}$,

for l = 1,2....m+1.

Proof:-

Consider $P(X_{kn} - X_{1n} \geq m/ X_{1n} =i)$

$= P(X_{kn} \geq m+i, X_{1n} =i)/ P(X_{1n} =i)$

Where $P(X_{kn} \geq m+i, X_{1n} =i)$ = P(Smallest X = i and atleast (n-k+1) X's ≥ m+i)

$= \sum_{s=1}^{k-1} \sum_{r=n-k+1}^{n-s}$ P(exactly S X's = i, exactly r X's>m+i and (n-s X's are in [i+1,m+i-1])

$$= \sum_{s=1}^{k-1} \sum_{r=n-k+1}^{n-s} \frac{n!}{S!(n-r-s)!} [G(i)– G(i+1)]^s G^r(m+i)$$

$$.[G(i+1) – G(m+i)]^{n-s-r}$$

Now letting G(i+1)/G(i) = q_i, i=1,2..., we have,

$$P(X_{kn} - X_{1n} \geq m/ X_{1n} =i) = [q^{-n}(i)-1]^{-1}. \sum_{s=1}^{k-1} \binom{n}{s}$$

$$[q^{-1}(i)-1]^{-s} \sum_{r=n-k+1}^{n-s} \binom{n-s}{r} \left[\prod_{j=1}^{m-1} q(i+j)\right]^r \left[1-\prod_{j=1}^{m-1} q(i+j)\right]^{n-s-r}$$

$$=[q^{-n}(i)-1]^{-1} \sum_{s=1}^{k-1} \binom{n}{s} [q^{-1}(i)-1]^{-s} \frac{(n-s)!}{(n-k)!(k-s-1)!} \int_0^{p_{im}} x^{n-k}(1-x)^{k-s-1} dx$$

Where $p_{im} = \left[\prod_{j=1}^{m-1} q(i+j)\right]$. If q(i) ≡q, then the above probability is free of i. However, if the conditional probability is free of i, then setting i=1 and i=2 and equating two equations, we obtain,

$$\sum_{s=1}^{k-1} \binom{n}{s}\binom{n-s}{n-k} (k-s) \int_{q^{m-2}.q(m+1)}^{q^{m-1}} x^{n-k}(1-x)^{k-s-1} dx = 0$$

Since all the terms in the preceding summation are non – negative, each has to be equal to zero and consequently q(m+1) = q. Now, induction will completes the proof.

Corollary 3.1.3.1

If m = 1, then by considering the complementary event. We infer that X_{1n} and the event {$X_{kn} - X_{1n} \geq m$} are independent if and only if $G(l) = q^{l-1}$, for l = 1,2...provided G(2) = q. For the next result, we need the following lemma.

Lemma 3.1.3.1

Let X be a discrete random variable having ($a_1,a_2...a_N$) for its set of possible values, where without loss of generality, we assume that a1< a2<.....<a_N. Furthermore, set G(l) = P(X≥a_l). Then X is degenerate if and only if for some arbitrarily fixed k (2 ≤ k ≤n), $P(X_{kn} = X_{1n}) = 1$.

Proof

If X is degenerate the result is trivially true. If X is non – degenerate, write

$$P(X_{kn} = X_{1n}) = \sum_{l=1}^{N} P(X_{kn} = X_{1n} = a_l)$$

$$= \sum_{l=1}^{N} \sum_{r=k}^{n} \binom{n}{r} [G(l) – G(l+1)]^r. G^{n-r}(l+1)$$

And assume that there is positive probability mass on two points only, namely a_i and a_j. Then

G(l) = 1 for l≤i, 0<G(i+1) = ...=G(j) <1, and G(l) = 0 for l>j. Hence, $P(X_{kn} = X_{1n}) = 1$ implies that,

$\sum_{r=k}^{n} \binom{n}{r} [G(l) – G(l+1)]^r. G^{n-r}(l+1) = 1$.That is since G(i) = 1,

$$\frac{(n)!}{(k-1)!(n-k)!} \int_0^{1-G(i+1)} x^{k-1}(1-x)^{n-k} dx = 1.$$

Hence, $\int_{1-G(i+1)}^{1} x^{k-1}(1-x)^{n-k} dx = 0$

Which implies that 1 – G(i+1) = 1 or G(i+1) = 0. This is a contradiction.

Theorem 3.1.3.2:-

Let {1,2,...N} denote the set of possible values for X where N may be finite or infinite. Then X_{1n} and the event $\{X_{kn} - X_{1n} \geq m\}$ are independent for some arbitrarily fixed k ($2 \leq k \leq n$) if and only if either X is degenerate or $G(l) = q^{l-1}$, for l = 1,2...provided G(2) = q.

Proof:-

First, let us assume that N is finite. Then due to the hypothesis, we have,

$P(X_{kn} - X_{1n} \geq 1, X_{1n} = i) = P(X_{kn} - X_{1n} \geq 1)\, P(X_{1n} = i)$

LHS $= P(X_{kn} \geq 1+i, X_{1n} = i)$ = P(smallest X=i, atleast (n-k+1) X's ≥ 1+i)

$$= \sum_{s=1}^{k-1} \text{P(exactly S X's = i, (n-s) X's} \geq 1+i)$$

$$= \sum_{s=1}^{k-1} \frac{(n)!}{(s)!(n-s)!} [G(i) - G(i+1)]^s .\, G^{n-s}(i+1)$$

Hence we have,

$$\sum_{s=1}^{k-1} \binom{n}{s} [G(i) - G(i+1)]^s .\, G^{n-s}(i+1) = P(X_{kn} - X_{1n} \geq 1)\, [G^n(i) - G^n(i+1)] \text{ for } i=1,2.$$

Let i = N and noting that G(l) = 0 for all l>N, we have,

$0 = P(X_{kn} - X_{1n} \geq 1)\, G^n(N)$. Since $G(N) \neq 0$, $P(X_{kn} - X_{1n} \geq 1) = 0$. That is

$P(X_{kn} - X_{1n} = 0) = 1$ which implies that X is degenerate by lemma 3.1.3.1.Hence N is infinite and the proof is complete by setting m = 1 in theorem 3.1.3.1.

Theorem 3.1.3.3:-

For some arbitrarily fixed k ($2 \leq k \leq n$), X_{1n} and the event $\{X_{kn} - X_{1n} = m\}$ are independent for some m≥1 if and only if $G(l) = q^{l-1}$, for l = 1,2......m+2.

Proof:-

Consider

$P(X_{kn} - X_{1n} = m / X_{1n} = i) = [P(X_{kn} \geq m+i, X_{1n} = i) - P(X_{kn} \geq m+i+1, X_{1n} = i)] / P(X_{1n} = i)$

Now using the proof of theorem 3.1.3.1, we have

$$\text{LHS} = \sum_{s=1}^{k-1} \binom{n}{s} [q^{-1}(i)-1]^{s-n} \int_{p_{i,m+1}}^{p_{im}} x^{n-k} (1-x)^{k-s-1} dx$$

From the hypothesis the right hand side expression is free of i, set i = 1 and 2 and equating both sides, we obtain,

$$\sum_{s=1}^{k-1} \binom{n}{s} [q^{-1}-1]^s \left\{ \int_{p_{2,m+1}}^{p_{2,m}} x^{n-k}(1-x)^{k-s} dx - \int_{p_{1,m+1}}^{p_{1,m}} x^{n-k}(1-x)^{k-s} dx \right\} = 0$$

Now since q(l) = q for l=1.....m+1. $p_{1,m} = q^{m-1}$, $p_{1,m+1} = q^m$, $p_{2,m} = q^{m-1}$, $p_{2,m+1} = q^{m+1}.q(m+2)$.

Consequently, we have

$$\sum_{s=1}^{k-1} \binom{n}{s} [q^{-1}-1]^s \int_{q^{m-1}.q(m+2)}^{q^m} x^{n-k} (1-x)^{k-s} dx = 0$$

Since each term is positive, it follows that q(m+2) = q. Now induction will completes the proof.

Theorem 2.1.3.4:-

For some arbitrarily fixed k ($1 \leq k \leq n-1$), X_{1n} and the event

$\{X_{k+1,n} - X_{k,n} \geq m, X_{k,n} = X_{1n}\}$ are independent for some m≥1 if and only if $G(l) = q^{l-1}$, for l = 1,2......provided q(l) = q (0<q<1) for l = 1,2..........m.

Proof:-

$P(X_{k+1,n} - X_{k,n} \geq m, X_{k,n} = X_{1n} / X_{1n} = i) = P(X_{k+1,n} \geq m+i, X_{kn} = X_{1n} = i) / P(X_{1n} = i)$

= P(exactly (n-k) X's ≥m+I and exactly k X's =i)/ $P(X_{1n} = i)$

$$= \binom{n}{k} [G(i) - G(i+1)]^k .\, G^{n-k}(m+i)\, [G^n(i) - G^n(i+1)]^{-1}$$

$$= \binom{n}{k} [1-q(i)]^k . \left[\prod_{h=0}^{m-1} q(i+h)\right]^{n-k} [1-q^n(i)]^{-1}$$

q(l) = q implies that the above quantity is free of i. If the above quantity is free of i, set i = 1 and 2and equating the quantities, we obtain,

$q^{m-1}.q(m+1) = q^m$. i.e., q(m+1) = q. Now inductions complete the proof.

THEOREM 2.1.3.5:-

For some arbitrarily fixed k ($1 \leq k \leq n-1$), X_{1n} and the event

$\{X_{k+1,n} - X_{k,n} = m, X_{k,n} = X_{1n}\}$ are independent for some m≥0 if and only if $G(l) = q^{l-1}$, for l = 1,2......provided

q(l) = q (0<q<1) for l = 1,2..........m+1.

PROOF:-

Write $P(X_{k+1,n} = m+i, X_{kn} = i, X_{1n} = i) / P(X_{1n} = i) = \dfrac{n!}{k!(n-k)!}$

$[G(i) - G(i+1)]^k [G(m+i) - G(m+i+1)].$

$G^{n-k-1}(m+i)\, [G^n(i) - G^n(i+1)]^{-1}$

$$= \binom{n}{k}[1-q(i)]^k.\left[\prod_{h=0}^{m-1} q(i+h)\right]^{n-k-1} [1-q(i+m)]\,[1-q^n(i)]^{-1}$$

q(l) = q implies that the above quantity is free of i. If the above quantity is free of i, set i = 1 and 2 and equating the quantities, we obtain,

$q^{m-1}.q(m+2) = q^m$. i.e., q(m+2) = q. Now induction completes the proof.

The following result gives another property enjoyed by the geometric distribution.

Theorem 3.3.1.6:-

If $G(i) = q^{i-1}$ for i=1,2....then

(i) X_{1n}andtheevent{$X_{k+1,n}$ - $X_{k,n} \geq$ m, $X_{k,n} \geq m$ }aremutually independent for all k (1 ≤ k ≤n-1), and all m≥1

(ii). X_{1n} and the event {$X_{k+1,n}$ - $X_{k,n}$= m} are mutually independent for all k (1 ≤ k ≤n-1), and all m≥0.

Proof:-

Consider

$P(X_{k+1,n} - X_{k,n} \geq m / X_{1n} = i) = P(X_{k+1,n} \geq m+i, X_{kn} = X_{1n} = i)$

$$+ \sum_{s=1}^{\infty} P(X_{k+1,n} \geq m+s+i, X_{kn} = s+i, X_{1n} = i)]/P(X_{1n} = i)$$

Now, $P(X_{k+1,n} \geq m+i, X_{kn} = X_{1n} = i)$ = P(exactly k X's =i and exactly (n-k)X's ≥m+i)

$$= \binom{n}{k} [G(i) - G(i+1)]^k G^{n-k}(m+i)$$

And for s≥1,

$P(X_{k+1,n} \geq m+s+i, X_{kn} = s+i, X_{1n} = i) = k(k-1) \binom{n}{k} [G(i) - G(i+1)][G(s+i)-G(s+i+1)][G(s+i)-G(s+i+1)]^{k-2} G^{n-k}(m+s+i)$

Thus $P(X_{k+1,n} - X_{k,n} \geq m / X_{1n} = i) = \binom{n}{k} [1-q^n(i)]^{-1}.[1-q(i)$

$$\left[\prod_{j=0}^{m-1} q(i+j)\right]^{n-k} \{(1-q(i))^{k-1} + k(k-1) \sum_{s=1}^{\infty} \left[\prod_{j=0}^{s-1} q(i+j)\right]$$

$$.[1-q(s+i)] \left[1 - \prod_{j=0}^{s} q(i+j)\right]^{k-2} \left[\prod_{j=m}^{m+s-1} q(i+j)\right]^{n-k}$$

Now, one can easily see that if q(i) = q for i = 1,2 then the right hand side expression is free of i. Result (ii) follows upon noting that .

$P(X_{k+1,n} - X_{k,n} = m/ X_{1n} = i) = [P(X_{k+1,n} \geq m+i / X_{1n} = i) - P(X_{k+1,n} - X_{k,n} \geq m/, X_{1n} = i)]$ and each quantity on right side is free of i.

3.2. Characterization Of Exponential Distribution Based On Order Statistics:-

3.2.1 .Introduction:-

Here we discuss M.S.Srivastava's characterization of the exponential distribution. The purpose of this note is to add one more existing set of theorems to characterize the exponential distribution whose probability density function is given by

f(x) = (1/σ) exp(-(x –θ) /σ), x >θ, σ>0. (3.1.2a)

A characterization of the exponential distribution:-

The following theorem characterizes the exponential distribution.

Theorem 3.2.1:-

Let F be an absolutely continuous distribution function of the random variable X with F(θ) = 0, θ>0,and with probability density function f(x). Let $X_1 < X_2 < \ldots\ldots < X_n$ denote the order statistics of a random sample of size n from this distribution. Then, in order that $X_{m+1} - X_m$ and X_m for fixed m, 1≤ m<n be independent, it is necessary and sufficient that the random variable X has the exponential distribution.

Proof:-

Necessity:-

Let $X_m = U$ and $X_{m+1} = V$. Then the pdf of U is,

$$\frac{n!}{(m-1)!(n-m)!}(F(u))^{m-1}(1-F(u)^{n-m} f(u)$$

And the joint pdf of U and V is,

$$\frac{n!}{(m-1)!(n-m-1)1}(F(u))^{m-1}(1-F(v))^{n-m-1} f(u)f(v)$$

Consequently the conditional pdf of V/U = u is,

$$f(V/U=u) = \frac{\frac{n!}{(m-1)!(n-m-1)!}(F(u))^{m-1}(1-F(u))^{n-m-1} f(u).f(v)}{\frac{n!}{(m-1)!(n-m)!}(F(u))^{m-1}(1-F(u))^{n-m}.f(u)}$$

$$= (n-m)\frac{(1-F(v))^{n-m-1}}{(1-F(u))^{n-m}}.f(v)$$

(2.2.1b)

It is our desire to show that if V-U and U are independent, then X has the exponential distribution. Because of the independence of V-U and U, E(V-U) = E(V-U/U=u) and is free of u. Thus,

E(V-U)=E(V-U/U=u)= $(n-m)\int_u^\infty (v-u)\left[\frac{(1-F(v))^{n-m-1}}{(1-F(u))^{n-m}}\right]f(v)dv$ (3.2.1c)

And is free of u. Differentiating the above equation with respect to u, we have with probability one,

$$0 \neq -\int_u^\infty \frac{(1-F(v))^{n-m-1}}{(1-F(u))^{n-m}}.f(v)dv + \frac{(n-m)f(u)}{(1-F(u))}.E(V-U)$$

Since, E(V-U) is independent of v and the first expression on the right is constant, we have,

$$\frac{f(u)}{1-F(u)} = \beta \qquad (3.2.1d)$$

Where β is a constant different from zero. Consequently,

$$1-F(u) = e^{-\beta u+d} \qquad (3.2.1e)$$

and $f(u) = \beta e^{-\beta u+d}$

Where d is a constant of integration. But since f is a probability density over the range θ to ∞.

It follows that d = βθ and β>0,hence,

$$f(u) = \beta e^{-\beta(x-\theta)} \qquad x>\theta, \beta>0 \quad (3.2.1f)$$

Therefore X has the exponential distribution.

Sufficiency:-

Now, suppose that the distribution of X is given by (3.2.1f). Then the joint density of U and V is,

$$f(u,v) = \beta^2 \frac{n!}{(m-1)!(n-m-1)!}.\left[1-e^{-\beta(u-\theta)}\right]^{m-1}.e^{-(n-m)\beta(v-\theta)-\beta(u-\theta)}$$

Consequently the joint density of Z=V-U and W=U is,

$$f(w,z) = \beta^2 \frac{n!}{(m-1)!(n-m-1)!}.\left[1-e^{-\beta(w-\theta)}\right]^{m-1}.e^{-(n-m+1)\beta(w-\theta)}.e^{-(n-m)\beta z}$$

, z>0, w>θ.

Hence Z and W are independent. Then the density of Z is given by,

$$(n-m)\beta.e^{-(n-m).\beta.z} \quad , z>0$$

And the density of W is given by,

$$\beta.\frac{n!}{(m-1)!(n-m)!}.\left[1-e^{-\beta(w-\theta)}\right]^{m-1}.e^{-(n-m+1)\beta(w-\theta)}, \quad w > \theta$$

3.2.2 Introduction:-

Here we discuss a characteristic property of the exponential distribution, which was given by M.Ashanullah. Let X be a random variable whose probability density function is,

$$f_\theta(x) = \begin{cases} \theta^{-1}e^{-x/\theta}, x>0 \\ 0 \quad \textit{otherwise} \end{cases} \qquad (3.2.2a)$$

Suppose $X_1, X_2....X_n$ is a random sample of size n from a population with density f and let $X_{1,n} < X_{2,n} <......< X_{n,n}$ be the associated order statistics. In this paper we will give a characterization of the exponential distribution that requires X and (n – i) $(X_{i+1,n} - X_{i,n})$ to be identically distributed for some i and n, 1≤ i < n.

Notation And Result:-

Let F be the distribution function of non – negative random variable and

let $\overline{F}$ (x) = 1-F(x), for x≥ 0.we will call F is 'New Better than Used' (NBU) if

$\overline{F}(x+y) \leq \overline{F}(x).\overline{F}(y)$, x,y ≥ 0, and F is 'New Worse than Used' if

$\overline{F}(x+y) \geq \overline{F}(x).\overline{F}(y)$, x,y ≥ 0. We will say that F belongs to the class C, if F is either NBU or NWU.

Trheorem 3.2.2:-

Let X be a non – negative random variable having an absolutely continuous (with respect to lebesque measure) distribution function F that is strictly increasing on (0,∞). Then the following properties are equivalent.

(a) X has an exponential distribution with density as given in (2.2.2a)
(b) For some i and n 1≤ i < n, the statistics (n – i) $(X_{i+1,n} - X_{i,n})$ and X are identically distributed and F belongs to class C.

Proof:-

It is well known that (a)⇒ (b) (see galambos (1975). So we prove only that (a)⇒ (b).

From the density of $Y_i = (X_{i+1,n} - X_{i,n})$, it follows that Z = (n-i) Y_i has the density,

$$f_Z(z) = L(n,i).\int_0^\infty (F(u))^{i-1}.(1-F(u+z(n-i)^{-1}))^{n-i-1}.f(u).f(u+z(n-i))^{-1}du/(n-i)$$

(2.2.2b)

Where $L(n,i) = \frac{n!}{(i-1)!(n-i-1)!}$. By using the hypothesis f_z = f and writing

$(n-i)/L(n,i)=\int_0^\infty (F(u))^{i-1}.(1-F(u))^{n-i}.f(u).du$. It follows that

$$f(z)=\left[\frac{1}{\int_0^\infty (F(u))^{i-1},(1-F(u))^{n-i}.f(u).du}\right].\int_0^\infty (F(u))^{i-1}.(1-F(u+z(n-i)^{-1}))^{n-i-1}.f(u).f(u+z(n-i)^{-1})du$$

$\Rightarrow$

$$\int_0^\infty (F(u))^{i-1}.f(u)\left[f(z).(1-F(u))^{n-i}-(1-F(u+z(n-i)^{-1}))^{n-i-1}.f(u+z(n-i)^{-1}\right]du=0$$

$$\int_0^\infty (F(u))^{i-1}.f(u).g(u,z)du=0 \text{ for all z.} \quad (3.2.2c)$$

Whereg(u,z)=

$$f(z).(1-F(u))^{n-i}-(1-F(u+z(n-i)^{-1}))^{n-i-1}.f(u+z(n-i)^{-1}$$

Integrating equation (3.2.2c) with respect to z from 0 to z_1, and interchanging the order of integration (which is permitted here), we get,

$$\int_0^\infty (F(u))^{i-1}.f(u)\left[\int_0^{z_1} f(z).(1-F(u))^{n-i}-(1-F(u+z(n-i)^{-1}))^{n-i-1}.f(u+z(n-i)^{-1}\right]du=0$$

$$\int_0^\infty (F(u))^{i-1}.(1-F(u))^{n-i}.f(u)G(u,z_1)du=0 \text{ for all z} \quad (3.2.2d)$$

Where

$$G(u,z_1)=\left[\frac{(1-F(u+z(n-i))^{n-i}}{(1-F(u))^{n-i}}\right]-(1-F(z_1))$$

Now if F is NBU, then for any integer k>0, $\overline{F}(x/k)\geq(\overline{F}(x))^{1/k}$, so G(0,$Z_1$) $\geq$ 0.Thus if (2.2.2c) holds, it must be G(0,Z_1) $\equiv$ 0,similarly if F is NWU, then G(0,Z_1) $\leq$ 0, and hence for (3.2.2d), to be true G(0,Z_1) $\equiv$ 0.writing G(0,Z_1) in terms of F, we get,

1-F(Z_1) = (1-F(Z_1(n-i)))$^{\text{n-i}}$, for all Z_1 (3.2.2e)

Substituting $\overline{F}$ (x) = 1-F(x) and n-i = k, we get from ((3.2.2e),

$$\overline{F}(z/k)=(\overline{F}(z))^{1/k},$$

for all Z_1>0 and some integers k>0. (3.2.2f)

The solution of (2.2.2f) is for k>1,

$$\overline{F}(z_1)=1-F(z_1)=e^{-\lambda_1 z_1}$$

for some λ_1>0 and all z>0 (2.2.2g)

If k=1, then

$G(u,Z_1)=(\overline{F}(u+Z_1))(\overline{F}(u)^{-1}-\overline{F}(Z_1)$ and (3.2.2d) gives,

$$\int_0^\infty (F(u))^{n-2}.f(u).\overline{F}(u)\left[\overline{F}(u+z_1.(\overline{F}(u)^{-1}-\overline{F}(z_1)\right]du=0$$

for all z_1 and with F$\in$C.

This means $\overline{F}(u+z_1).(\overline{F}(u))^{-1}=\overline{F}(z_1)$. So again we get,

$\overline{F}(z_1)=1-F(z_1)=e^{-\lambda_1 z_1}$ for some λ_1>0 and all z>0

3.2.3. Introduction:-

Here we discuss the characterization of exponential distribution, which was given by Sukhatme(1937).Let X be a random variable whose probability density function is given for some θ>0,

$f_\theta(x) = \theta e^{-\theta x}$, x >θ, θ>0.

Suppose X_1,X_2....X_n is a random sample of size n from a population with density f and let $X_{(1)}$<$X_{(2)}$<........<$X_{(n)}$, be the associated order statistics. In this note, we will give a characterization of the exponential distribution that requires D_k= (n-k+1) {$X_{(k)}$ – $X_{(k-1)}$} to be identically distributed with pdf f. The following theorem gives the characterization of the exponential distribution.

Theorem:-

Let $X_{(1)}$,$X_{(2)}$........$X_{(n)}$, be order statistics based on a random sample from $f_\theta(x) = \theta e^{-\theta x}$, x >θ, θ>0.. Define D_k= (n-k+1) {$X_{(k)}$ – $X_{(k-1)}$} , where $X_{(0)}$=0.Then D_1,D_2....D_n are iid random variables with pdf, $\theta e^{-\theta x}$, x >θ.

Proof:-

The joint pdf of $X_{(1)}$,$X_{(2)}$........$X_{(n)}$ is

$$g(x_1.....x_n)=n!\prod_{i=1}^{n}\theta e^{-\theta.x_i} \qquad 0<x_1<....<x_n<\infty$$

$$= n!\,\theta^n\, e^{-\theta\sum_1^n x_i} \qquad 0<x_1<....<x_n<\infty$$

Define d_k= (n-k+1) {$x_{(k)}$ – $x_{(k-1)}$}, k=1,2.....n. then the Jacobin is,

$$J^{-1}=\left|\left(\left(\frac{\partial d_i}{\partial x_j}\right)\right)\right| = n! \quad \Rightarrow J = (1/n!).$$

Consider, $\sum_{i=1}^{n} x_i = x_1 + x_2 + ... + x_n$ =

$x_1 + \overline{(x_2 - x_1} + x_1) + ... + \overline{(x_n - x_{n-1}} + \overline{x_{n-1} - x_{n-2}} + ... + \overline{x_2 - x_1} + x_1)$

= $n.x_1+(n-1).(x_2-x_1)+.... ....+(x_n-x_{n-1})$

Therefore $\sum_{i=1}^{n} x_i = \sum_{i-1}^{n} di$ .Then the joint pdf of $D_1..D_n$ is,

$h(d_1,d_2......d_n)=n!.\theta^n e^{-\theta\sum_1^n x_i}$ (1/n!) $0<d_1<....<d_n<\infty$

$= \prod_{i=1}^{n} \theta.e^{-\theta\sum_{k=1}^{n} d_k}$ $0<d_1<....<d_n<\infty$

Therefore $D_1.....D_n$ are iid exponential with pdf f.

Corollary:-

(i). $nX_{(1)}$ follows exponential with parameter θ.

(ii).The statistic $\sum_{i=2}^{n}(X_{(i)} - X_{(1)})$ follows gamma distribution with (α, n-1).

PROOF:-

(i) $nX_{(1)} = D_1$. From the above theorem, D_1 has exponential distribution.

(ii) Consider $\sum_{i=2}^{n}(X_{(i)} - X_{(1)})$ = $X_{(2)} - X_{(1)}+X_{(3)}-X_{(1)}+........+X(n) - X_{(1)}$

= $X_{(2)} - X_{(1)} + X_{(3)}-X_{(2)} + X_{(2)} - X_{(1)} + ... + X(n) - X_{(n-1)} + X_{(n-1)} + X_{(n-2)} + X_{(2)} - X_{(1)}$ = $(n-1)(X_{(2)} - X_{(1)}) + (n-2)(X_{(3)}-X_{(2)})+...+ (X(n) - X_{(n-1)})$

(iii) We know that, $nX_{(1)} = D_1$ and $\sum_{i=2}^{n}(X_{(i)} - X_{(1)})$ = $D_2+D_3+.....+D_n$

Since $D_1.....D_n$ are independent. Therefore $D_1 \coprod \sum_{i=2}^{n} D_i$ .which implies that $X_{(1)}$ and $\sum_{i=2}^{n}(X_{(i)} - X_{(1)})$ are independent.

3.3. Characterizations By Record Values:-

Let $X_1,X_2,...X_n$ be a sequence of iid random variable with continuous distribution function F. Define a sequence of record times U(n), n≥1 as follow:

$U(1) =1, U(n) = \min \{j : j>U(n-1), X_j > X_{U(n-1)}\}, n>1$

Let $X_{u(n)}$ be upper record values, n=1,2..... Let $X_{U(n+1)}, X_{U(n+2)}$ ...... be the next observation that come after $X_{U(n)}$.It is not difficult to prove that $X_{U(n)}, X_{U(n+1)}, X_{U(n+2)}$ ...... are mutually independent and $X_{U(n)+k}$ has the same distribution F for any k=1,2.....Let us define the following random variable for a give n

$$\eta_n(i) = \begin{cases} 1 \text{ if } X_{U(n)+i} < X_{U(n)}, i=1,2,.... \\ 0 \text{ if } X_{U(n)+i} \geq X_{U(n)}. \end{cases}$$

As a consequence of theorem 3.1 we have the following.

Theorem 3.1.1:-

Let X be a non-negative random variable having continuous distribution function F satisfying inf{ x : F(x)>0 }=0,then the following statements are equivalent

(a).X has an exponential distribution with density as given in (3.1.2)

For some n>1,E($(\xi_n(1)) = E(\eta_n(1))$

And F is either NBU or NWU

PROOF:-

It is not difficult to see that if F is continuous distribution function F, then,

$$P\{X_{U(n)+1} < X_{U(n)}\} = 1- \frac{1}{2^n}$$

By assumption of the theorem $F \in \Im_a$, and

$$P\{X_{n+1} < X_1 + X_2 +....+X n\} = 1- \frac{1}{2^n}$$

And from theorem 3.1 F is exponential.

Tata (1969) proved that if Y_1,Y_2 ....are iid r.v's with the distribution function $F_0(x) = 1-\exp(-x)$, x>0 then, the random variables

$Z_1 = Y_1, Z_2 = Yu_{(2)} -Y_1........Z_n =Y_{u(n)} -Y_{u(n)}$

are independent and $P\{Z_n \leq x\} = F_0 (x)$ (n = 1,2,....).It is all true that for iid r.v's if the differences $X_{U(n)} - X_{U(n-1)}$, n≥2 are independent, then the population is exponential .Using this, we get, $Y_{U(n)} \overset{d}{=} Y_1 + Y_2 + ...+Y_n$, n=1,2,...

where $\overset{d}{=}$ denote equality in distribution

Theorem 4.1.2:-

Let $X_1,X_2,...X_n$ be a sequence of iid random variable with continuous distribution function F satisfying inf{ x : F(x)>0 }=0 then the following statements are equivalent

(a) X has an exponential distribution with density as given in (4.1.2)
(b) For some n>1,

$$X_1+X_2+........+X_n \overset{d}{=} X_{U(n)} \text{ and } F \in \Im_1$$

Proof:-

It is clear that (a)$\Rightarrow$(b). Now we will prove other implication.

Let $X_1+X_2+\dots+X_n \overset{d}{=} X_{U(n)}$ and $F \in \Im_1$.

Consider

$P\{F_0(Y_1+Y_2+\dots+Y_n)\} \le P\{Y_1+Y_2+\dots+Y_n \le F_0^{-1}(t)\}$ (3.1.10)

$$= \frac{1}{(n-1)!}\int_0^{-\ln(1-t)} e^{-x}x^{n-1}dx = \frac{1}{(n-1)!}\int_0^{t}[\ln\frac{1}{1-x}]^{n-1}dx$$

We have $P\{X_1+X_2+\dots+X_n \le u\} = P\{X_{U(n)} \le u\}$

$$= \frac{1}{(n-1)!}\int_0^{-\ln(1-F(u))} e^{-x}x^{n-1}dx \quad (3.1.11)$$

Taking $F(u) = t$, $u = F^{-1}(t)$ in (4.1.11), we obtain

$$P\{F(X_1+X_2+\dots+X_n) \le F^{-1}t\} = \frac{1}{(n-1)!}\int_0^{-\ln(1-t)} e^{-x}x^{n-1}dx \quad (3.1.12)$$

From (3.1.10) and (3.1.12), we have,

$P\{F(X_1+X_2+\dots+X_n) \le t\} = P\{F_0(Y_1+Y_2+\dots+Y_n) \le t\}$, $t \in [0,1]$ (3.1.13)

Then one can show that,

$P\{X_{n+1} < X_1 + X_2 + \dots + X_n\} = P\{Y_{n+1} < Y_1 + Y_2 + \dots + Y_n\}$

$$= 1 - \frac{1}{2^n} \quad (3.1.14)$$

Infactonehas,

$P\{X_{n+1} < X_1 + X_2 + \dots + X_n\} = \int\int\dots\int F(u_1 + u_2 + \dots + u_n)\, dF(u_1)\dots dF(u_n) = E[F\{X_1 + X_2 + \dots + X_n\}]$

$= \int x\, dp\{F(X_1 + X_2 + \dots + X_n) \le x\}$

$P\{Y_{n+1} < Y_1+Y_2+\dots+Y_n\} = \int x\, dp\{F_0(X_1 + X_2 + \dots + X_n) \le x\}$.

Then by using (3.1.13) one can obtain (3.1.14). From theorem 3.1, F is exponential, which concludes the proof.

4. CONCLUSION

In this study we conclude that the characterization of Geometric and Exponential Distribution based on Order statistics has a wide applications and they have a unique characterizations.

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